GABOR-TYPE FRAMES FROM GENERALIZED WEYL-HEISENBERG GROUPS

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ABSTRACT. We present in this paper a construction for Gabor-type frames built out of generalized Weyl-Heisenberg groups. These latter are obtained via central extensions of groups which are direct products of locally compact abelian groups and their duals. Our results generalize many of the results, appearing in the literature, on frames built out of the Schrödinger representation of the standard Weyl-Heisenberg group. In particular, we obtain a generalization of the result in [1], in which the product ab determines whether it is possible for the Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$. As a particular example of the theory, we study in some detail the case of the generalized Weyl-Heisenberg group built out of the d-dimensional torus. In the same spirit we also construct generalized shift-invariant systems.

1. Introduction

Recently, a generalization of the Weyl-Heisenberg group has been presented in [12, 13]. Such a generalized Weyl-Heisenberg group is the central extension of the direct product of a locally compact abelian group G with its dual group $\hat{G}$. By analogy with the standard Weyl-Heisenberg group, it is then possible to construct Schrödinger-type representations in these general situations, which are again continuous, unitary and irreducible. Since a Gabor system can be considered as the orbit of a discrete subset of the Weyl-Heisenberg group, under the Schrödinger action, this paper presents a generalization of such a system using a discrete subset of the generalized Weyl-Heisenberg group. There exists an abundance of frame-related results, in the literature, for Gabor frames. A good sampling of these may be found, for example, in [1, 4, 7, 9, 10, 15]. This paper presents extensions of several of these results, in particular those dealing with the boundedness and invertibility of the frame operator, to generalized Gabor systems. Specifically, we refer to Theorems 3.6, 4.2, 4.3 and 5.3 below. We ought to also mention that generalized Weyl-Heisenberg groups have also been looked at for other and related studies. A good reference is the work edited by G. Feichtinger and Werner Kozek [7] and in particular, Chapter 7 of that book.

The rest of this paper is organized as follows: Section 2 sets out some definitions and a few known results related to Gabor frames. Section 3 presents the definition of the generalized Weyl-Heisenberg group and generalizations, in this setting, of some results mentioned in the previous section. Section 4 is devoted to the study of a special case: the generalized Weyl-Heisenberg group associated to the torus $\mathbb{T}^d$. In Theorem 4.2 we work out the analogue on $L^2(\mathbb{T}^d)$, for this group, of the (by now) standard result that the product ab puts a condition on the system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$. This result is further generalized, to any LCA group, in Theorem 4.3. Finally, we define a generalized shift-invariant system on $L^2(G)$ and present some results associated to it.

Date: February 7, 2008.
1991 *Mathematics Subject Classification.* 81R30.
Key words and phrases. Weyl Heisenberg group, Schrödinger representation, Gabor frames.

2. Preliminaries

We begin by giving the central definitions and some necessary and sufficient conditions for a standard Gabor system to be a frame.

2.1. **Some definitions.**

A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is called a Bessel sequence if there exists a constant $B>0$ such that

$$\sum_{k=1}^{\infty} \mid \langle f, f_k\rangle \mid^2 \leq B \parallel f \parallel^2, \quad \forall f \in \mathcal{H}.$$

A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded bijective operator.

(i) A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A \parallel f \parallel^2 \leq \sum_{k=1}^{\infty} \mid \langle f, f_k\rangle \mid^2 \leq B \parallel f \parallel^2, \quad \forall f \in \mathcal{H}.$$

The numbers A and B are called frame bounds.

(ii) A frame is tight if we can choose $A = B$ as frame bounds.

The following Lemma will be useful in the sequel.

Lemma 2.1. *Let $\{f_k\}_{k=1}^{\infty}$, be a frame. Then the following are equivalent:*

(i) $\{f_k\}_{k=1}^{\infty}$ *is tight.*

(ii) $\{f_k\}_{k=1}^{\infty}$ *has a dual of the form $g_k = Cf_k$, for some constant $C > 0$.*

2.2. **Weyl-Heisenberg frame.**

Let x, w be real numbers. The unitary operators defined on $L^2(\mathbb{R})$ by $T_x f(y) = f(y-x)$, and $E_w f(y) = e^{2\pi w.y} f(y)$, are called translation and modulation operators, respectively.

A Weyl-Heisenberg frame, or synonymously, a Gabor frame is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, where $a, b > 0$ and $g \in L^2(\mathbb{R})$ is a fixed function. Explicitly,

$$E_{mb}T_{na}g(x) = e^{2\pi mbx} g(x - na).$$

The function g is called the window function or the frame generator. For an exhaustive list of papers dealing with such frames we refer to the monograph [7].

Our main results in this paper will consist of generalizations of the following four theorems for standard Gabor or Weyl-Heisenberg frames.

Theorem 2.2. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following holds:*

(i) If $ab > 1$, then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is not a frame for $L^2(\mathbb{R})$.

(ii) If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then

$ab = 1 \Leftrightarrow \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Riesz basis.

In [14], there is the following stronger result than (i) : when $ab > 1$, the family $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can not even be complete in $L^2(\mathbb{R})$. The assumption $ab \leq 1$ is not enough

for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame, even if $g \neq 0$. For example, if $a \in [\frac{1}{2}, 1[$, the set of functions $\{E_m T_{na}\chi_{[0,\frac{1}{2}]}\}_{m,n\in\mathbb{Z}}$ is not complete in $L^2(\mathbb{R})$ and cannot form a frame.

Proposition 8.3.2 in [3] gives a necessary condition for a Gabor system to be a frame. Sufficient conditions for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$ have been known since 1988, the basic insight being provided by Daubechies [5]. A slight improvement was proved in [10]. Later, Ron and Shen [15] were able to give a complete characterization of Gabor frames, spelled out in the next theorem. Given $g \in L^2(\mathbb{R})$, consider the matrix-valued function

$$M(x) = \{g\,(x - na - m/b)\}_{m,n\in\mathbb{Z}}\,. \tag{2.1}$$

The matix $M(x)M^*(x)$ is positive.

Theorem 2.3. *$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B if and only if*

$$bA\mathbb{I} \leq M(x)M^*(x) \leq bB\mathbb{I},\ a.e., \tag{2.2}$$

where $\mathbb{I}$ is the identity operator on $\ell^2(\mathbb{Z})$.

This theorem is a special case of the following characterization [3] of a shift-invariant system to be a frame.

Recall that if $\{g_m\}_{m\in I}$ is a collection of functions in $L^2(\mathbb{R})$, the shift-invariant system generated by $\{g_m\}_{m\in I}$ and some $a \in \mathbb{R}$ is the collection of functions $\{g_m(. - na)\}_{m\in I n\in\mathbb{Z}}$. Usually we will set $I = \mathbb{Z}$.

Given a shift-invariant system $\{g_m(. - na)\}_{n,m\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, define the matrix-valued function $H(\nu)$, $\nu \in \mathbb{R}$, by

$$H(\nu) \;=\; \left(\hat{g}_m(\nu - k/a)\right)_{k,m\in\mathbb{Z}}\,, \quad a.e.\,, \tag{2.3}$$

$\hat{g}$ denoting the Fourier transform of g. The following theorem then contains a generalization of Theorem 2.3 to any shift-invariant system $\{g_m(. - na)\}_{n,m\in\mathbb{Z}}$ for $L^2(\mathbb{R})$:

Theorem 2.4. *With the above setting, the following hold:*

(i) *$\{g_{n,m}\}$ is a Bessel sequence with upper bound B if and only if $H(\nu)$, for almost all ν, defines a bounded operator on $l^2(\mathbb{Z})$ of norm at most $\sqrt{aB}$.*

(ii) *$\{g_{n,m}\}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B if and only if*

$$aA\mathbb{I} \leq H(\nu)H^*(\nu) \leq aB\mathbb{I}, \quad a.e.\ \nu. \tag{2.4}$$

(iii) *$\{g_{n,m}\}$ is a tight frame for $L^2(\mathbb{R})$ if and only if there is a constant $c > 0$ such that*

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m(\nu)}\hat{g}_m(\nu + k/a) = c\delta_{k,0}, \quad k \in \mathbb{Z},\ a.e.\ \nu. \tag{2.5}$$

In case (2.5) is satisfied, the frame bound is $A = c/a$.

(iv) *Two shift-invariant systems $\{g_{n,m}\}$ and $\{h_{n,m}\}$, which form Bessel sequences, are dual frames if and only if*

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m(\nu)}\hat{h}_m(\nu + k/a) = a\delta_{k,0}, \quad k \in \mathbb{Z},\ a.e.\ \nu. \tag{2.6}$$

Theorem 2.3 is difficult to apply. However, the condition on the matrix $M(x)M^*(x)$ is in particular satisfied if it is diagonal dominant. This leads to a sufficient condition [1] for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$.

Theorem 2.5. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ and suppose that*

$$B := \frac{1}{b} \sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-k/b)} \mid < \infty. \tag{2.7}$$

Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with upper frame bound B. If also

$$A := \frac{1}{b} \inf_{x\in[0,a]} \left[\sum_{n\in\mathbb{Z}} \mid g(x-na) \mid^2 - \sum_{0\neq k\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-k/b)} \mid \right] > 0. \tag{2.8}$$

Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B.

3. Generalized Weyl-Heisenberg Group

Definition 3.1. Let G be a locally compact abelian (LCA) group, $\hat{G}$ its dual group, μ and ν their Haar measures, respectively. Let $\mathbb{T}$ be the unit circle and put $H_G = G \times \hat{G} \times \mathbb{T}$. For (g_1, w_1, z_1) and (g_2, w_2, z_2) in H_G, define the following composition:

$$(g_1, w_1, z_1).(g_2, w_2, z_2) = \big(g_1 g_2, w_1 w_2, z_1 z_2 w_2(g_1)\big). \tag{3.1}$$

H_G is closed under this action, which is also associative. Equipped with this product, H_G is a group, called the generalized Weyl-Heisenberg group, associated with G. This group is nonabelian, locally compact, and unimodular [16], with invariant measure $d\mu d\nu d\theta$ (where $z = e^{i\theta}$).

A uniform lattice in G is a discrete subgroup K of G such that G/K is compact. For a uniform lattice K in G, $Ann(K)$, denotes the annihilator of K, *i.e.*, $Ann(K) := \{\gamma \in \hat{G} \ : \gamma(k) = 1, \forall\, k \in K\}$.

By Lemma 24.5 of [11], we know that $Ann(K) = \widehat{G/K}$, so that $Ann(K)$ is a discrete subgroup of $\hat{G}$.

Let $\pi : H_G \longrightarrow U\Big(L^2(G)\Big)$ be the Schrödinger representation of H_G, which is a unitary, irreducible representation, given explicitly by

$$\Big(\pi(x, \gamma, z)g\Big)(t) = z\gamma(t)g(tx^{-1}), \tag{3.2}$$

for all $(x, \gamma, z) \in H_G$ and almost all $g \in L^2(G)$. In [12], frames of $L^2(G)$ of the type

$$\Big\{\Theta^g_{(k,\gamma)} = \Big(\pi(k, \gamma, 1)g\Big) : \ (k, \gamma) \in K \times Ann(K)\Big\}, \tag{3.3}$$

where K is a uniform lattice in G, have been studied. In this case, taking $G = \mathbb{R}$ and $K = a\mathbb{Z}$ and defining the dual pairing in the usual way:

$$\xi(x) = e^{2\pi i x\xi}, \tag{3.4}$$

we obtain $Ann(K) = \dfrac{1}{a}\mathbb{Z}$. Thus, the Gabor system defined by (3.3) is

$$\Big\{e^{\frac{2\pi i m x}{a}} g(x-na)\Big\}_{m,n\in\mathbb{Z}}, \tag{3.5}$$

which is a particular case ($ab = 1$) of the standard Gabor system:

$$\big\{e^{2\pi i m b x} g(x-na)\big\}_{m,n\in\mathbb{Z}}. \tag{3.6}$$

In this paper, we study frames for $L^2(G)$ of the form

$$\left\{\Theta^g_{(k_1,\gamma_2)} = \Big(\pi(k_1,\gamma_2,1)g\Big) : (k_1,\gamma_2) \in K_1 \times Ann(K_2)\right\}, \tag{3.7}$$

where K_1 and K_2 are two lattices in G. Such a frame is clearly a generalization of a Gabor frame, because if we take $K_1 = a\mathbb{Z}$ and $K_2 = \frac{1}{b}\mathbb{Z}$ and use the same dual pairing as above, we get exactly the standard Gabor system (3.6).

Definition 3.2. The set defined by

$$\left\{\Theta^g_{(k_1,\gamma_2)} : (k_1,\gamma_2) \in K_1 \times Ann(K_2)\right\} \tag{3.8}$$

will be called the generalized Gabor system for $L^2(G)$ associated to the uniform lattices K_1 and K_2, and the window function g.

The following Lemmata are essential for this work:

Lemma 3.3. *Let $f, g \in L^2(G)$. and K_1 and K_2 be two uniform lattices in G. Then, for any $k_2 \in K_2$, the series*

$$\sum_{k_1\in K_1} f(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})} \tag{3.9}$$

converges absolutely for almost all $x \in G$ and the function

$$x \longmapsto \sum_{k_1\in K_1} \mid f(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})} \mid \in\; L^1(G/K_1), \tag{3.10}$$

Proof. Since $f \in L^2(G)$ and $T_{k_2}g \in L^2(G)$, then $f.T_{k_2}g \in L^1(G)$. But,

$$\int_{G/K_1} \sum_{k_1\in K_1} \mid f(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})} \mid dx = \int_G \mid f(x)\overline{g(xk_2^{-1})} \mid dx < \infty, \tag{3.11}$$

by Hölder's inequality. This implies that $\sum_{k_1\in K_1} \mid f(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})} \mid < \infty$, *a.e.* □

Lemma 3.4. *Let $f, g \in L^2(G)$ and K_1, K_2 be two uniform lattices in G. For a fixed $k_1 \in K_1$, consider the function $F_{k_1} \in L^1(G/K_2)$, defined by*

$$F_{k_1}(x) = \sum_{k_2\in K_2} f(xk_2^{-1})\overline{g(xk_1^{-1}k_2^{-1})}. \tag{3.12}$$

Then, for any $(k_1,\gamma_2) \in K_1 \times Ann(K_2)$, we have

$$\langle f|\Theta^g_{(k_1,\gamma_2)}\rangle = \int_{G/K_2} \overline{\gamma_2(x)} F_{k_1}(x)dx. \tag{3.13}$$

Proof.

$$\begin{aligned}
\langle f|\Theta^g_{(k_1,\gamma_2)}\rangle &= \int_G \overline{\gamma_2(x)} f(x)\overline{g(xk_1^{-1})}dx \\
&= \int_{G/K_2} \sum_{k_2\in K_2} \overline{\gamma_2(xk_2^{-1})} f(xk_2^{-1})\overline{g(xk_1^{-1}k_2^{-1})}dx \\
&= \int_{G/K_2} \overline{\gamma_2(x)} \sum_{k_2\in K_2} f(xk_2^{-1})\overline{g(xk_1^{-1}k_2^{-1})}dx \\
&= \int_{G/K_2} \overline{\gamma_2(x)} F_{k_1}(x)dx. \qquad (3.14)
\end{aligned}$$

□

Let us now present the generalization of the WH-frame identity (see [3], Lemma 8.4.3) for an arbitrary LCA group G. Our generalization appears in Lemma 3.5 below. Consider the function H_{1_G} defined by

$$H_{1_G}(x) = \sum_{k_1\in K_1} \mid g(xk_1^{-1}) \mid^2 \quad . \qquad (3.15)$$

Lemma 3.5. *Let $f, g \in L^2(G)$ and K_1, K_2 be two uniform lattices in G. Suppose that f is a bounded measurable function with compact support and that the function H_{1_G} defined by (3.15) is bounded. Then*

$$\begin{aligned}
& \sum_{k_1\in K_1} \sum_{\gamma_2\in Ann(K_2)} \mid \langle f|\Theta^g_{(k_1,\gamma_2)}\rangle \mid^2 \qquad (3.16) \\
= & \mid G/K_2 \mid \int_G \mid f(x) \mid^2 \sum_{k_1\in K_1} \mid g(xk_1^{-1}) \mid^2 dx \\
+ & \mid G/K_2 \mid \sum_{1_G\neq k_2\in K_2} \int_G \overline{f(x)} f(xk_2^{-1}) \sum_{k_1\in K_1} g(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})}dx \ ,
\end{aligned}$$

where $\mid G/K_2 \mid$ denotes the measure of G/K_2.

Proof. From Theorem 4.26 in [8], we conclude that the set of functions $\{\mid G/K_2 \mid^{-\frac{1}{2}} \gamma_2\}_{\gamma_2\in Ann(K_2)}$, is an orthonormal basis for $L^2(G/K_2)$. By Parseval's theorem, we have

$$\sum_{\gamma_2\in Ann(K_2)} \mid \int_{G/K_2} \overline{\gamma_2(x)} F_{k_1}(x)dx \mid^2 = \mid G/K_2 \mid \int_{G/K_2} \mid F_{k_1}(x) \mid^2 dx. \qquad (3.17)$$

which implies that

$$
\begin{aligned}
(3.18)\qquad & \sum_{k_1\in K_1}\sum_{\gamma_2\in Ann(K_2)} \mid \langle f|\Theta^g_{(k_1,\gamma_2)}\rangle \mid^2 \\
&= \sum_{k_1\in K_1}\sum_{\gamma_2\in Ann(K_2)} \mid \int_{G/K_2} \overline{\gamma_2(x)} F_{k_1}(x)dx \mid^2 \\
&= \mid G/K_2 \mid \sum_{k_1\in K_1} \int_{G/K_2} \mid F_{k_1}(x) \mid^2 dx \\
&= \mid G/K_2 \mid \sum_{k_1\in K_1} \int_{G/K_2} F_{k_1}(x) \sum_{k_2\in K_2} \overline{f(xk_2^{-1})} g(xk_2^{-1}k_1^{-1})dx \\
&= \mid G/K_2 \mid \sum_{k_1\in K_1} \int_{G} F_{k_1}(x)\overline{f(x)} g(xk_1^{-1})dx \\
&= \mid G/K_2 \mid \sum_{k_1\in K_1} \int_{G} \overline{f(x)} g(xk_1^{-1}) \sum_{k_2\in K_2} f(xk_2^{-1})\overline{g(xk_2^{-1}k_1^{-1})}dx \\
&= \mid G/K_2 \mid \int_{G} \mid f(x) \mid^2 \sum_{k_1\in K_1} \mid g(xk_1^{-1}) \mid^2 dx \\
&+ \mid G/K_2 \mid \sum_{1_G\neq k_2\in K_2} \int_{G} \overline{f(x)} f(xk_2^{-1}) \sum_{k_1\in K_1} g(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})}dx.
\end{aligned}
$$

□

The following is a generalization of Theorem 2.5 to any LCA group G.

Theorem 3.6. *Let K_1 and K_2 be two uniform lattices of the LCA group G. Let $g \in L^2(G)$ such that:*

$$
B := \mid G/K_2 \mid \sup_{x\in G/K_1} \sum_{k_2\in K_2} \mid \sum_{k_1\in K_1} g(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})} \mid < \infty. \tag{3.19}
$$

Then $\{\Theta^g_{k_1,\gamma_2}\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)}$ is a Bessel sequence with upper frame bound B. If also

$$
A := \mid G/K_2 \mid \inf_{x\in G/K_1} \left[\sum_{k_1\in K_1} \mid g(xk_1^{-1}) \mid^2 - \sum_{1_G\neq k_2\in K_2} \mid \sum_{k_1\in K_1} g(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})} \mid \right] > 0 \,, \tag{3.20}
$$

then $\{\Theta^g_{k_1,\gamma_2}\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)}$ is a frame for $L^2(G)$ with bounds A, B.

Proof. For $k_2 \in K_2$, fixed, define the function H_{k_2} by

$H_{k_2}(x) = \sum_{k_1\in K_1} T_{k_1}g(x)\overline{T_{k_1k_2}g(x)}$. We have:

$$
\begin{aligned}
\sum_{1_G\neq k_2\in K_2} \mid T_{k_2^{-1}} H_{k_2}(x) \mid &= \sum_{1_G\neq k_2\in K_2} \mid T_{k_2^{-1}} \sum_{k_1\in K_1} T_{k_1}g(x)\overline{T_{k_1k_2}g(x)} \mid \\
&= \sum_{1_G\neq k_2\in K_2} \mid \sum_{k_1\in K_1} T_{k_1k_2^{-1}}g(x)\overline{T_{k_1}g(x)} \mid . \qquad (3.21)
\end{aligned}
$$

Replacing k_2 by k_2^{-1}, we have

$$\begin{aligned}
\sum_{1_G \neq k_2 \in K_2} \mid T_{k_2^{-1}} H_{k_2}(x) \mid &= \sum_{1_G \neq k_2 \in K_2} \mid T_{k_2^{-1}} \sum_{k_1 \in K_1} T_{k_1} g(x) \overline{T_{k_1 k_2} g(x)} \mid \\
&= \sum_{1_G \neq k_2 \in K_2} \mid \sum_{k_1 \in K_1} T_{k_1 k_2^{-1}} g(x) \overline{T_{k_1} g(x)} \mid . \\
&= \sum_{1_G \neq k_2 \in K_2} \mid \sum_{k_1 \in K_1} T_{k_1} g(x) \overline{T_{k_1 k_2} g(x)} \mid . \\
&= \sum_{1_G \neq k_2 \in K_2} \mid H_{k_2}(x) \mid . \qquad (3.22)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mid \sum_{1_G \neq k_2 \in K_2} \int_G \overline{f(x)} f(x k_2^{-1}) \sum_{k_1 \in K_1} g(x k_1^{-1}) \overline{g(x k_1^{-1} k_2^{-1})} dx \mid \qquad (3.23) \\
&= \mid \sum_{1_G \neq k_2 \in K_2} \int_G \overline{f(x)} T_{k_2} f(x) H_{k_2}(x) dx \mid, \\
&\leq \sum_{1_G \neq k_2 \in K_2} \int_G \mid f(x) \mid . \mid T_{k_2} f(x) \mid . \mid H_{k_2}(x) \mid dx \\
&\quad \text{(using the Cauchy-Schwarz inquality twice: with respect} \\
&\quad \text{to the integral and the sum)} \\
&\leq \int_G \mid f(x) \mid^2 . \sum_{1_G \neq k_2 \in K_2} \mid H_{k_2}(x) \mid dx.
\end{aligned}$$

By Lemma 3.5, we have

$$\begin{aligned}
&\sum_{k_1 \in K_1} \sum_{\gamma_2 \in Ann(K_2)} \mid \langle f | \Theta^g_{(k_1, \gamma_2)} \rangle \mid^2 \\
&\leq \mid G/K_2 \mid \int_G \left(\mid f(x) \mid^2 \left[H_{1_G}(x) + \sum_{1_G \neq k_2 \in K_2} \mid \sum_{k_1 \in K_1} T_{k_1} g(x) \overline{T_{k_1 k_2} g(x)} \mid \right] \right) dx \\
&= \mid G/K_2 \mid \int_G dx \mid f(x) \mid^2 \sum_{k_2 \in K_2} \mid \sum_{k_1 \in K_1} T_{k_1} g(x) \overline{T_{k_1 k_2} g(x)} \mid \\
&\leq B \mid\mid f \mid\mid^2 .
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\sum_{k_1 \in K_1} \sum_{\gamma_2 \in Ann(K_2)} \mid \langle f | \Theta^g_{(k_1, \gamma_2)} \rangle \mid^2 \\
&\geq \mid G/K_2 \mid \int_G \left(\mid f(x) \mid^2 \left[H_{1_G}(x) - \sum_{1_G \neq k_2 \in \mathcal{K}_2} \mid \sum_{k_1 \in K_1} T_{k_1} g(x) \overline{T_{k_1 k_2} g(x)} \mid \right] \right) dx \\
&\geq A \parallel f \parallel^2 .
\end{aligned}$$

Since the frame conditions hold for all f in a dense subspace of $L^2(G)$, it is true for any element of $L^2(G)$. $\square$

Remark 3.7. The above result is more general than, and is in fact an extension to other classes of groups, of the results in [1, 10]. By taking $G = \mathbb{R}$, $K_1 = a\mathbb{Z}$, and $K_2 = \frac{1}{b}\mathbb{Z}$, we recover Theorem 2.5.

4. Frames on the torus $\mathbb{T}^d$

Let $G = \mathbb{T}^d$ be the torus in d dimensions. Let $N_i, M_i \in \mathbb{N}^*,\ i = 1, 2, ..., d$, be $2d$ positive integers. For simplicity, we adopt the following notation in this section:

Let $\underline{\mathfrak{n}} = (n_1, ..., n_d)$, $\underline{\mathfrak{N}} = (N_1, ..., N_d)$ and $\underline{\mathfrak{M}} = (M_1, ..., M_d)$. Set $(\underline{\mathfrak{n}}, \underline{\mathfrak{N}}) = \left(\frac{n_1}{N_1}, \frac{n_2}{N_2}, ..., \frac{n_d}{N_d}\right)$ and $(\underline{\mathfrak{m}}, \underline{\mathfrak{M}}) = \left(\frac{m_1}{M_1}, \frac{m_2}{M_2}, ..., \frac{m_d}{M_d}\right)$ and consider the following two uniform lattices in $\mathbb{T}^d$:

$$\mathcal{K}_1^{\mathfrak{N}} = \{(\underline{\mathfrak{n}}, \underline{\mathfrak{N}}) : n_i = 0, 1, ..., N_i - 1, \text{ for } i = 1, ..., d\} \tag{4.1}$$

and

$$\mathcal{K}_2^{\mathfrak{M}} = \{(\underline{\mathfrak{m}}, \underline{\mathfrak{M}}) : m_i = 0, 1, ..., M_i - 1, \text{ for } i = 1, ..., d\}. \tag{4.2}$$

Using these, we form the sets

$$\mathbb{T}^d/\mathcal{K}_1^{\mathfrak{N}} = \left[0, \frac{1}{N_1}\right] \times ... \times \left[0, \frac{1}{N_d}\right] \equiv \Delta_1,$$

$$\mathbb{T}^d/\mathcal{K}_1^{\mathfrak{M}} = \left[0, \frac{1}{M_1}\right] \times ... \times \left[0, \frac{1}{M_d}\right] \equiv \Delta_2$$

and

$$Ann(\mathcal{K}_2^{\mathfrak{M}}) = \left\{\gamma_{\underline{k}}(\underline{x}) = e^{2\pi i \sum_{j=1}^d M_j k_j x_j};\quad \underline{k} = (k_1, ..., k_d) \in \mathbb{Z}^d\right\}.$$

Note that $\left\{\left(\prod_{j=1}^d M_i\right)^{\frac{1}{2}} \gamma\right\}_{\gamma \in Ann(\mathcal{K}_2^{\mathfrak{M}})}$ is an orthonormal basis of $L^2(\Delta_2)$, and we have:

Corollary 4.1. Let $g \in L^2(\mathbb{T}^d)$, and $N_i, M_i,\ i = 1, 2, \ldots, d$, be $2d$ positive integers such that:

$$\begin{aligned} B :&= \frac{1}{\prod_{j+1}^d M_j} \sup_{\underline{x} \in \Delta_1} \sum_{(\underline{\mathfrak{m}}, \underline{\mathfrak{M}}) \in \mathcal{K}_2^{\mathfrak{M}}} \Big| \sum_{(\underline{\mathfrak{n}}, \underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{M}}} g\left([\underline{x} - (\underline{\mathfrak{n}}, \underline{\mathfrak{N}})]\right) \\ &\times \overline{g\left([\underline{x} - (\underline{\mathfrak{n}}, \underline{\mathfrak{N}}) - (\underline{\mathfrak{m}}, \underline{\mathfrak{M}})]\right)} \Big| < \infty. \end{aligned} \tag{4.3}$$

Then $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}}, \underline{\mathfrak{N}})} g\right\}_{((\underline{\mathfrak{n}}, \underline{\mathfrak{N}}), \underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ is a Bessel sequence for $L^2(\mathbb{T}^d)$ with upper bound B. If also

$$\begin{aligned} A := \frac{1}{\prod_{j+1}^d M_j} \inf_{\underline{x} \in \Delta_1} \Bigg[&\sum_{(\underline{\mathfrak{n}}, \underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{M}}} \left| g\left([\underline{x} - (\underline{\mathfrak{n}}, \underline{\mathfrak{N}})]\right) \right|^2 \\ - \sum_{\underline{\mathfrak{o}} \neq (\underline{\mathfrak{m}}, \underline{\mathfrak{M}}) \in \mathcal{K}_2^{\mathfrak{M}}} &\Big| \sum_{(\underline{\mathfrak{n}}, \underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{M}}} g\left([\underline{x} - (\underline{\mathfrak{n}}, \underline{\mathfrak{N}})]\right) \overline{g\left([\underline{x} - (\underline{\mathfrak{n}}, \underline{\mathfrak{N}}) - (\underline{\mathfrak{m}}, \underline{\mathfrak{M}})]\right)} \Big| \Bigg] > 0, \end{aligned} \tag{4.4}$$

then $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}}, \underline{\mathfrak{N}})} g\right\}_{((\underline{\mathfrak{n}}, \underline{\mathfrak{N}}), \underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ is a frame for $L^2(\mathbb{T}^d)$ with bounds A, B.

We observe immediately that the canonical commutation relations,

$$T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}\gamma_{\underline{k}} = e^{2\pi i \sum_{j=1}^{d} M_j (\underline{\mathfrak{n}},\underline{\mathfrak{N}})_j k_j}\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} \ ,$$

hold in this case. Indeed, for $g \in L^2(\mathbb{T}^d)$ we see that,

$$\begin{aligned} \left(T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}\gamma_{\underline{k}} g\right)(\underline{x}) &= T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}\left(e^{2\pi i \sum_{j=1}^{d} M_j x_j k_j} g\left(\underline{x}\right)\right) = e^{2\pi i \sum_{j=1}^{d} M_j \left(x_j - \frac{n_j}{N_j}\right) k_j} g\left([\underline{x} - (\underline{\mathfrak{n}},\underline{\mathfrak{N}})]\right) \\ &= e^{-2\pi i \sum_{j=1}^{d} \frac{M_j n_j k_j}{N_j}} \gamma_{\underline{k}}(\underline{x}) g\left([\underline{x} - (\underline{\mathfrak{n}},\underline{\mathfrak{N}})]\right) \\ &= e^{-2\pi i \sum_{j=1}^{d} M_j (\underline{\mathfrak{n}},\underline{\mathfrak{N}})_j k_j} \gamma_{\underline{k}}(\underline{x}) T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\left(\underline{x}\right) . \end{aligned}$$

It will be useful, for the purposes of the next section, to note that the frame operator for a frame $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ commutes with the corresponding modulation and translation operators.

Lemma 4.1. *Let $g \in L^2(\mathbb{T}^d)$ and let $N_i, M_i,\ i = 1, 2, \ldots, d,$ be $2d$ positive integers such that $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ is a frame for $L^2(\mathbb{T}^d)$. If S is the corresponding frame operator then,*

$$ST_{\left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)}\gamma_{\underline{k}_0} = T_{\left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)}\gamma_{\underline{k}_0} S, \quad \text{for all} \quad \underline{k_0} \in \mathbb{Z}^d, \text{ and all } \left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right) \in \mathcal{K}_1^{\mathfrak{N}}.$$

Proof. Let $f \in L^2(\mathbb{T}^d)$. We know that

$$S(f) = \sum_{\underline{k} \in \mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}} \langle f \mid \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\rangle \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g \ , \tag{4.5}$$

so that,

$$\begin{aligned} \left(S\gamma_{\underline{k}_0} T_{\left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)}\right) f &= \sum_{\underline{k} \in \mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}} \langle \gamma_{\underline{k}_0} T_{\left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)} f \mid \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\rangle \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g \\ &= \sum_{\underline{k} \in \mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}} \langle f \mid T_{\left(-\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)} \gamma_{-\underline{k_0}} \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\rangle \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g \\ &= \sum_{\underline{k} \in \mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}} \langle f \mid e^{2\pi i \sum_{j=1}^{d} M_j \left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)_j (k_j - k_{0j})} \gamma_{\underline{k} - \underline{k_0}} T_{\left(\underline{\mathfrak{n} - \mathfrak{n}_o},\underline{\mathfrak{N}}\right)} g\rangle \\ &\times \ \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g \\ &= \sum_{\underline{k} \in \mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}} \langle f \mid e^{2\pi i \sum_{j=1}^{d} M_j \left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)_j k_j} \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\rangle \\ &\times \ \gamma_{\underline{k + k_0}} T_{\left(\underline{\mathfrak{n} + n_0},\underline{\mathfrak{N}}\right)} g \\ &= \sum_{\underline{k} \in \mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}} \langle f \mid e^{2\pi i \sum_{j=1}^{d} M_j \left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)_j k_j} \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\rangle \\ &\times \ e^{2\pi i \sum_{j=1}^{d} M_j \left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)_j k_j} \gamma_{\underline{k_0}} T_{\left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)} \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g \\ &= \sum_{\underline{k} \in \mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}} \langle f \mid \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\rangle \gamma_{\underline{k_0}} T_{\left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)} \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g \\ &= \left(\gamma_{\underline{k_0}} T_{\left(\underline{\mathfrak{n}_0},\underline{\mathfrak{N}}\right)} S\right) f. \end{aligned}$$

□

4.1. **Necessary condition for having frames on the torus $\mathbb{T}^d$.** We derive, in the next theorem, conditions for the existence of frames on $L^2(\mathbb{T}^d)$, which will be analogues of the conditions imposed by the product ab, in Theorem 2.2, for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a Gabor frame for $L^2(\mathbb{R})$.

We start by fixing a certain partition of $\mathbb{T}^d$. For $k=1,...,d$, let $i_k \in \{0,1,...,N_k-1\}$. For a $d-$tuplet $(i_1,i_2,...,i_d)$, of such i_k, let us define the subset $\Gamma_{(i_1,i_2,...,i_d)}$ by

$$\Gamma_{(i_1,i_2,...,i_d)} = \left[\frac{i_1}{N_1},\frac{i_1+1}{N_1}\right] \times .... \times \left[\frac{i_d}{N_d},\frac{i_d+1}{N_d}\right] \ . \tag{4.6}$$

It is easy to see that these subsets have the following properties:

$$\cup_{i_k=0,...,N_k-1;k=1,...,d}\,\Gamma_{(i_1,i_2,...,i_d)} = \mathbb{T}^d \ , \tag{4.7}$$

$$\Gamma_{(i_1,i_2,...,i_d)} \cap \Gamma_{(i'_1,i'_2,...,i'_d)} = \emptyset \ a.e. \text{ if } (i_1,i_2,...,i_d) \neq (i'_1,i'_2,...,i'_d) \ , \tag{4.8}$$

$$T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}\left(\Gamma_{(i_1,i_2,...,i_d)}\right) \cap \Gamma_{(i_1,i_2,...,i_d)} = \emptyset, \ a.e. \tag{4.9}$$

for all $(\underline{\mathfrak{o}},\underline{\mathfrak{N}}) \neq (\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}$ and all $(i_1,i_2,...,i_d)$.

Let

$$\mathcal{F}(\mathbb{T}^d) = \left\{f \in L^2(\mathbb{T}^d) \ : \ \exists (i_1,i_2,...,i_d) \ \text{ and } \ supp(f) \subset \Gamma_{(i_1,i_2,...,i_d)}\right\}$$

Then, by virtue of (4.7),

$$\mathcal{F}(\mathbb{T}^d) \quad \text{is dense in} \quad L^2(\mathbb{T}^d) \ , \tag{4.10}$$

and by (4.9),

$$\forall f \in \mathcal{F}(\mathbb{T}^d), f(\underline{x}).\overline{T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}f(\underline{x})} = 0, \tag{4.11}$$

for almost all $\underline{x} \in \mathbb{T}^d$ and all $(\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}$ and $(\underline{\mathfrak{o}},\underline{\mathfrak{N}}) \neq (\underline{\mathfrak{n}},\underline{\mathfrak{N}})$.

The following gives a necessary condition for having frame on $L^2(\mathbb{T}^d)$

Theorem 4.2. *Let $g \in L^2(\mathbb{T}^d)$, and let $N_i, M_i, \ i=1,2,\dots,d$, be $2d$ positive integers. Then the following hold:*

(i) *If* $\left(\prod_{i=1}^d M_i\right) > \left(\prod_{j=1}^d N_j\right)$, *then* $\left\{\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ *is a not a frame for* $L^2(\mathbb{T}^d)$.

(ii) *If* $\left\{\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ *is a frame for* $L^2(\mathbb{T}^d)$, *then* $\prod_{i=1}^d M_i = \prod_{j=1}^d N_j \Leftrightarrow \left\{\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ *is a Riesz basis.*

Proof. Let us assume that $\left\{\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{T}^d)$ with frame operator S. Then $\left\{S^{-\frac{1}{2}}\left(\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})}g\right)\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ is a tight frame for $L^2(\mathbb{T}^d)$ with frame

bounds 1. Let $f \in \mathcal{F}(\mathbb{T}^d)$. Using *Lemma* 3.5, *Lemma* 4.1, and (4.11), we have:

$$\begin{aligned}
\int_{\mathbb{T}^d} \mid f(\underline{x}) \mid^2 d\underline{x} &= \sum_{\underline{k}\in\mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})\in\mathcal{K}_1^{\mathfrak{N}}} \mid \langle f \mid S^{-\frac{1}{2}} \left(\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\right)\rangle \mid^2 \\
&= \sum_{\underline{k}\in Z^d} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})\in\mathcal{K}_1^{\mathfrak{N}}} \mid \langle f \mid \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} S^{-\frac{1}{2}} g\rangle \mid^2 \\
&= \frac{1}{\prod_{j=1}^d M_j} \int_{\mathbb{T}^d} \mid f(\underline{x}) \mid^2 \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})\in\mathcal{K}_1^{\mathfrak{M}}} \mid S^{-\frac{1}{2}} g\left([\underline{x} - (\underline{\mathfrak{n}},\underline{\mathfrak{N}})]\right) \mid^2 dx, \qquad (4.12)
\end{aligned}$$

which implies that

$$\sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})\in\mathcal{K}_1^{\mathfrak{M}}} \mid S^{-\frac{1}{2}} g\left([\underline{x} - (\underline{\mathfrak{n}},\underline{\mathfrak{N}})]\right) \mid^2 = \prod_{j=1}^d M_j, \quad a.e., \ in\ \mathbb{T}^d. \qquad (4.13)$$

Since

$$\left\{S^{-\frac{1}{2}} \left(\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\right)\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$$

is a tight frame, we have

$$\begin{aligned}
1 &\geq \| S^{-\frac{1}{2}} \left(\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\right) \|^2 \\
&= \int_{\mathbb{T}^d} \mid S^{-\frac{1}{2}} g(\underline{x}) \mid^2 d\underline{x} \\
&= \int_0^{\frac{1}{N_1}} \int_0^{\frac{1}{N_2}} \dots \int_0^{\frac{1}{N_d}} \sum_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})\in\mathcal{K}_1^{\mathfrak{M}}} \mid S^{-\frac{1}{2}} g\left([\underline{x} - (\underline{\mathfrak{n}},\underline{\mathfrak{N}})]\right) \mid^2 d\underline{x} \\
&= \int_0^{\frac{1}{N_1}} \int_0^{\frac{1}{N_2}} \dots \int_0^{\frac{1}{N_d}} \prod_{j=1}^d M_j d\underline{x} \\
&= \left(\prod_{j=1}^d M_j\right) \left(\prod_{j=1}^d N_j\right)^{-1},
\end{aligned}$$

which proves (i). In order to prove part (ii), let us assume that $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ is a Riesz Basis. Then $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} S^{-\frac{1}{2}} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ is a Riesz Basis having bounds $A = B = 1$, which implies that $\| S^{-\frac{1}{2}} g \|^2 = 1$. So we have $\prod_{j=1}^d M_j = \prod_{j=1} N_j$. For the second implication, let us assume that $\prod_{j=1}^d M_j = \prod_{j=1} N_j$. Then $\| \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} S^{-\frac{1}{2}} g \|^2 = 1$, for all $\underline{k} \in Z^d$ and all $(\underline{\mathfrak{n}},\underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}$. Thus, $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} S^{-\frac{1}{2}} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ is a an orthonormal Basis for $L^2(\mathbb{T}^d)$, and therefore,

$$\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d} = \left\{S^{\frac{1}{2}} \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} S^{-\frac{1}{2}} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d} \qquad (4.14)$$

is a Riesz basis. □

4.2. **Generalization to arbitrary LCA group.** We show next that the above theorem can be extended to any LCA group G, thus constituting a generalization of Theorem 2.2 to any such group. Before stating the result, we note two commutation relations.

For $(k_1, \gamma_2) \in K_1 \times Ann(K_2)$, we have

$$T_{k_1}\gamma_2 = \gamma_2\left(k_1^{-1}\right)\gamma_2 T_{k_1}. \tag{4.15}$$

Also, for $(k_1^0, \gamma_2^0) \in K_1 \times Ann(K_2)$, we have

$$S\gamma_2^0 T_{k_1^0} = \gamma_2^0 T_{k_1^0} S\ . \tag{4.16}$$

Indeed, for $f, g \in L^2(G)$,

$$\begin{aligned}
\left(S\gamma_2^0 T_{k_1^0}\right) f &= \sum_{k_1\in K_1}\sum_{\gamma_2\in Ann(K_2)} \langle \gamma_2^0 T_{k_1^0} f | \gamma_2 T_{k_1} g\rangle \gamma_2 T_{k_1} g \\
&= \sum_{k_1\in K_1}\sum_{\gamma_2\in Ann(K_2)} \langle f | T_{\left(k_1^0\right)^{-1}}\left(\gamma_2^0\right)^{-1}\gamma_2 T_{k_1} g\rangle \gamma_2 T_{k_1} g \\
&= \sum_{k_1\in K_1}\sum_{\gamma_2\in Ann(K_2)} \langle f | \left(\left(\gamma_2^0\right)^{-1}\gamma_2\right)(k_1^0)\left(\gamma_2^0\right)^{-1}\gamma_2 T_{k_1\left(k_1^0\right)^{-1}} g\rangle \gamma_2 T_{k_1} g \\
&= \sum_{\tilde{k}_1\in K_1}\sum_{\tilde{\gamma}_2\in Ann(K_2)} \langle f | \tilde{\gamma}_2(k_1^0)\tilde{\gamma}_2 T_{\tilde{k}_1} g\rangle \tilde{\gamma}_2(k_1^0)\gamma_2^0 T_{k_1^0}\tilde{\gamma}_2 T_{\tilde{k}_1} g\ ,
\end{aligned}$$

whence the result.

Theorem 4.3. *Let $g \in L^2(G)$, and K_1 and K_1 be two uniform lattices in G. Then, the following hold:*

(i) *If* $\frac{|G/K_1|}{|G/K_2|} > 1$, *then* $\{\Theta^g_{k_1,\gamma_2}\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)}$ *is a not a frame for $L^2(G)$.*
(ii) *If* $\{\Theta^g_{k_1,\gamma_2}\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)}$ *is a frame for $L^2(G)$, then* $\mid G/K_1 \mid = \mid G/K_2 \mid \Leftrightarrow \{\Theta^g_{k_1,\gamma_2}\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)}$ *is a Riesz basis.*

Proof. Let $\sigma : G/\mathcal{K}_1 \longrightarrow G$, be any section. For any $k_1 \in K_1$, let $\sigma_{k_1} = T_{k_1}\left(\sigma\left(G/\mathcal{K}_1\right)\right)$. We have:

(a) $\cup_{k_1\in\mathcal{K}_1}\sigma_{k_1} = G$
(b) $\sigma_{k_1}\cap\sigma_{k_2} = \emptyset$ *a.e. in* G, $\forall\, k_1 \neq k_2$.

Let

$$\mathcal{F}(G) = \left\{f \in L^2(G)\ :\ \exists k_1 \in \mathcal{K}_1 \quad \text{and} \quad supp(f) \subset \sigma_{k_1}\right\}$$

and assume that $\{\Theta^g_{k_1,\gamma_2}\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a frame for $L^2(G)$. Then, the set of vectors $\{\Theta^{S^{-\frac{1}{2}}g}_{k_1,\gamma_2}\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a tight frame with bounds 1. Let $f \in \mathcal{F}(G)$. Using Lemma 3.5, and the statements (a) and (b), we have:

$$\begin{aligned}
\int_G \mid f(\underline{x}) \mid^2 d\underline{x} &= \sum_{k_1\in\mathcal{K}_1}\sum_{\gamma_2 Ann(\mathcal{K}_2)} \mid \langle f \mid \gamma_2 T_{k_1} S^{-\frac{1}{2}} g\rangle \mid^2 \\
&= \mid G/K_2 \mid \int_G \mid f(\underline{x}) \mid^2 \sum_{k_1\in\mathcal{K}_1} \mid S^{-\frac{1}{2}} g\left(xk_1^{-1}\right) \mid^2 dx,
\end{aligned} \tag{4.17}$$

which implies that

$$\sum_{k_1\in\mathcal{K}_1} \mid S^{-\frac{1}{2}} g\left(xk_1^{-1}\right) \mid^2 = \mid G/K_2 \mid^{-1}, \quad \text{a.e. in} \quad G. \tag{4.18}$$

Since $\left\{S^{-\frac{1}{2}}\left(\gamma_2 T_{k_1} g\right)\right\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a tight frame, we have

$$\begin{aligned} 1 &\geq \| S^{-\frac{1}{2}}\left(\gamma_2 T_{k_1} g\right) \|^2 \\ &= \int_G \mid S^{-\frac{1}{2}} g(\underline{x}) \mid^2 d\underline{x} \\ &= \int_{G/\mathcal{K}_1} \sum_{k_1\in\mathcal{K}_1} \mid S^{-\frac{1}{2}} g\left(xk_1^{-1}\right) \mid^2 d\underline{x} \\ &= \int_{G/\mathcal{K}_1} \mid G/K_2 \mid^{-1} dx \\ &= \mid G/K_1 \mid . \mid G/K_2 \mid^{-1} = \frac{\mid G/K_1 \mid}{\mid G/K_2 \mid}, \end{aligned}$$

which proves (i). For part (ii) let assume that $\{\gamma_2 T_{k_1} g\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(K_2)}$ is a Riesz basis. Then the set of vectors $\left\{S^{-\frac{1}{2}}\left(\gamma_2 T_{k_1} g\right)\right\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a Riesz Basis having bounds $A = B = 1$, which implies that $\| S^{-\frac{1}{2}} g \|^2 = 1$. So, we have $\mid G/K_1 \mid = \mid G/K_2 \mid$. For the second implication, let assume that $\mid G/K_1 \mid = \mid G/K_2 \mid$, then $\| \gamma_2 T_{k_1} S^{-\frac{1}{2}} g \|^2 = 1$, for all $(k_1,\gamma_2) \in \mathcal{K}_1 \times Ann(\mathcal{K}_2)$. Then $\left\{\gamma_2 T_{k_1} S^{-\frac{1}{2}} g\right\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a an orthonormal Basis for $L^2(G)$, and therefore,

$$\{\gamma_2 T_{k_1} g\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)} = \left\{S^{\frac{1}{2}} \gamma_2 T_{k_1} S^{-\frac{1}{2}} g\right\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)} \tag{4.19}$$

is a Riesz basis. □

5. General shift-invariant systems

In this section, using an obvious generalization of the notion of a shift invariant system, defined in Section 2, we present a complete characterization of a generalized Gabor frame on $L^2(G)$, where G is any locally compact Abelian group. The result will be an extension of the result of Ron and Shen [15] on $L^2(\mathbb{R})$. Recall that for an LCA group the Fourier transform is a map $\mathcal{F}$ defined from $L^1(G) \longrightarrow C(\hat{G})$ by

$$(\mathcal{F}f)(\xi) := \hat{f}(\xi) = \int_G \overline{\langle x,\xi\rangle} f(x) dx. \tag{5.1}$$

This can be extended to a map $L^2(G) \longrightarrow L^2(\hat{G})$, satisfying the well-known Plancherel identity. The following properties of the Fourier transform will be required in the sequel:

$$\begin{aligned} \left(\widehat{T_y f}\right)(\xi) &= \int_G \overline{\langle x,\xi\rangle} f(y^{-1}x) dx = \int_G \overline{\langle yx,\xi\rangle} f(x) dx \\ &= \overline{\langle y,\xi\rangle} \hat{f}(\xi), \end{aligned} \tag{5.2}$$

$$\left(\widehat{\eta f}\right)(\xi) \;=\; \int_G \overline{\langle x,\xi\rangle}\langle x,\eta\rangle f(x)dx = \hat{f}\left(\eta^{-1}\xi\right). \tag{5.3}$$

Let $\{g_m\}_{m\in\mathbb{Z}}$ be a collection of functions in $L^2(G)$ and K_1 a uniform lattice in G. For $m \in \mathbb{Z}$ and $k_1 \in K_1$, consider the function $g_{k_1,m}$ defined on G by $g_{k_1,m}(x) = g_m(xk_1^{-1})$.

Lemma 5.1. *Let $\{g_{k_1,m}\}_{m\in\mathbb{Z};k_1\in K_1}$ and $\{h_{k_1,m}\}_{m\in\mathbb{Z};k_1\in K_1}$ be two shift invariant systems and assume that they are Bessel sequences. Then, for $e,\ f \in L^2(G)$, the function,*

$$\begin{aligned} P(e,f): \;\; G \;&\longrightarrow \mathbb{C} \\ x \;&\longmapsto \sum_{m\in\mathbb{Z}}\sum_{k_1\in K_1}\langle T_x e \mid g_{k_1,m}\rangle\langle h_{k_1,m} \mid T_x f\rangle \end{aligned} \tag{5.4}$$

is continuous and well defined on G/K_1. Its Fourier series in $L^2(G/K_1)$ is

$$P(e,f)(x) \;=\; \sum_{\gamma_1\in Ann(K_1)} c_{\gamma_1}\gamma_1(x)\ , \tag{5.5}$$

where,

$$c_{\gamma_1} \;=\; \mid G/K_1 \mid^{-1}\int_{\hat{G}} \hat{e}(\xi)\overline{\hat{f}(\xi\gamma_1)}\sum_{m\in\mathbb{Z}}\overline{\hat{g_m}(\xi)}\hat{h}_i(\xi\gamma_1)d\xi, \quad \gamma_1 \in Ann(K_1).$$

Proof. Using the Cauchy-Schwarz inequality and the fact that the sets $\{g_{k_1,m}\}_{m\in\mathbb{Z};k_1\in K_1}$ and $\{h_{k_1,m}\}_{m\in\mathbb{Z};k_1\in K_1}$ are Bessel sequences, we conclude that the series defined by $P(e,f)(x)$ converges absolutely. Also, for any $k_1 \in K_1$, we have $P(e,f)(xk_1) = P(e,f)(x)$, for almost all $x \in G$. Hence $P(e,f)$ is well defined as a function on G/K_1. For the determination of the Fourier coefficients, let assume that $e,\ f$ are continuous and have compact supports. Then the coefficients c_{γ_1}, with respect to $\{\gamma_1(x)\}_{\gamma_1\in Ann(K_1)}$, are given by

$$\begin{aligned} c_{\gamma_1} \;&=\; \mid G/K_1 \mid^{-1}\int_{G/K_1}\rho(e,f)(x)\overline{\gamma_1(x)}dx \\ &=\; \mid G/K_1 \mid^{-1}\sum_{m\in\mathbb{Z}}\sum_{k_1\in K_1}\int_{G/K_1}\langle T_x e \mid g_m\left(.k_1^{-1}\right)\rangle\langle h_m\left(.k_1^{-1}\right) \mid T_x f\rangle\overline{\gamma_1(x)}dx \\ &=\; \mid G/K_1 \mid^{-1}\sum_{m\in\mathbb{Z}}\int_G\langle T_x e \mid g_m\rangle\langle h_m \mid T_x f\rangle\overline{\gamma_1(x)}dx \\ &=\; \mid G/K_1 \mid^{-1}\sum_{m\in\mathbb{Z}}\int_G\langle T_x e \mid g_m\rangle\overline{\langle T_x f \mid h_m\rangle\gamma_1(x)}dx\ . \end{aligned} \tag{5.6}$$

For an arbitrary $\phi \in L^2(G)$,

$$\langle T_x e \mid \phi\rangle = \langle \mathcal{F}T_x e \mid \mathcal{F}\phi\rangle = \int_{\hat{G}}\overline{\xi(x)}\hat{e}(\xi)\overline{\hat{\phi}(\xi)}d\xi = \mathcal{F}\left(\hat{e}.\overline{\hat{\phi}}\right)(x). \tag{5.7}$$

The last equality is justified by identifying G and $\hat{\hat{G}}$ and adopting the natural dual pairing.

Also, using (5.7) and (5.2), we have:

$$\begin{aligned}
\langle T_x f \mid h_m\rangle \gamma_1(x) &= \gamma_1(x)\mathcal{F}\left(\hat{f}.\overline{\hat{h}_m}\right)(x) \\
&= \langle x, \gamma_1\rangle \mathcal{F}\left(\hat{f}.\overline{\hat{h}_m}\right)(x) = \overline{\langle x, \gamma_1^{-1}\rangle}\mathcal{F}\left(\hat{f}.\overline{\hat{h}_m}\right)(x) \\
&= \mathcal{F}\left(T_{\gamma_1^{-1}}\hat{f}.\overline{\hat{h}_m}\right)(x). \qquad (5.8)
\end{aligned}$$

Using (5.7) and (5.8) in (5.6), we have

$$\begin{aligned}
c_{\gamma_1} &= \mid G/K_1 \mid^{-1} \sum_{m\in\mathbb{Z}} \int_G \mathcal{F}\left(\hat{e}.\overline{\hat{g}_m}\right)(x)\overline{\mathcal{F}\left(T_{\gamma_1^{-1}}\hat{f}.\overline{\hat{h}_m}\right)}(x)dx \\
&= \mid G/K_1 \mid^{-1} \sum_{m\in\mathbb{Z}} \int_{\hat{G}} \left(\hat{e}.\overline{\hat{g}_m}\right)(\xi)\left[\overline{T_{\gamma_1^{-1}}\left(\hat{f}.\overline{\hat{h}_m}\right)}\right](\xi)d\xi \\
&= \mid G/K_1 \mid^{-1} \int_{\hat{G}} \hat{e}(\xi).\overline{\hat{f}(\xi\gamma_1)} \sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1)d\xi. \qquad (5.9)
\end{aligned}$$

□

We proceed to characterize the frame properties of shift-invariant systems for $L^2(G)$, where G is an arbitrary LCA group. Let $\{g_m\}_{m\in\mathbb{Z}}$, be a collection of functions in $L^2(G)$ and K_1 a uniform lattice in G. For $\xi \in \hat{G}$, consider the matrix valued function $H(\xi) = \{\hat{g}_{\gamma_1,m}(\xi) = \hat{g}_m\left(\xi\gamma_1^{-1}\right)\}_{m\in\mathbb{Z};\gamma_1\in Ann(K_1)}$.

Proposition 5.2. *Assume that the system $\{g_{\gamma_1,m}\}_{(\gamma_1,m)\in Ann(K_1)\times\mathbb{Z}}$, has finite upper frame bound B. Then, for almost all $\xi \in \hat{G}$, $H(\xi)$ defines a bounded linear operator from $\ell^2(\mathbb{Z})$ into $\ell^2\left(Ann(K_1)\right)$ with operator norm $\leq (\mid G/K_1 \mid B)^{1/2}$. Explicitly,*

$$\sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1)\beta_m \mid^2 \leq \mid G/K_1 \mid B \parallel \underline{\beta} \parallel^2, \qquad (5.10)$$

for almost all $\xi \in \hat{G}$ and $\underline{\beta} \in \ell^2(\mathbb{Z})$.

Proof. Since $\hat{G}/Ann\,(K_1)$ is compact, let $d\xi$ be the Haar measure on $\hat{G}/Ann\,(K_1)$ normalized so that $\mid \hat{G}/Ann\,(K_1) \mid = \frac{1}{|G/K_1|}$. Using the fact that $\widehat{\left(\frac{\hat{G}}{Ann(K_1)}\right)} = Ann\left(Ann(K_1)\right) = K_1$, we see that K_1 is an orthonormal basis of $L^2\left(\hat{G}/Ann\,(K_1), \mid G/K_1 \mid d\xi\right)$, where the action is defined in a natural way by $k_1(\xi) := \xi(k_1)$.

Let $\alpha_{k_1,m} \neq 0$ for only finitely many $(k_1, m) \in K_1 \times \mathbb{Z}$, and let

$$\alpha_m(\xi) = \sum_{k_1\in K_1} \alpha_{k_1,m}\overline{\xi(k_1)} = \sum_{k_1\in K_1} \alpha_{k_1,m}\overline{k_1(\xi)}. \qquad (5.11)$$

For any $\gamma_1 \in Ann\,(K_1)$, we have $\alpha_m(\xi\gamma_1) = \alpha_m(\xi)$. Thus, α_m is well defined as a function on $\hat{G}/Ann\,(K_1)$, and we have:

$$\int_{\hat{G}/Ann(K_1)} \sum_{\gamma_1 \in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \alpha_m(\xi)\hat{g}_m(\xi\gamma_1) \mid^2 d\xi = \int_{\hat{G}} \mid \sum_{m\in\mathbb{Z}} \alpha_m(\xi)\hat{g}_m(\xi) \mid^2 d\xi. \tag{5.12}$$

Using Parseval's theorem (or the fact that the Fourier transform is an unitary operator) and (5.2), we have:

$$\int_{\hat{G}} \mid \sum_{k_1\in K_1; m\in\mathbb{Z}} \alpha_{k_1,m}\xi(k_1)\hat{g}_m(\xi) \mid^2 d\xi = \parallel \sum_{k_1\in K_1; m\in\mathbb{Z}} \alpha_{k_1,m} g_{k_1,m} \parallel^2 . \tag{5.13}$$

Also, we have

$$\parallel \sum_{k_1\in K_1; m\in\mathbb{Z}} \alpha_{k_1,m} g_{k_1,m} \parallel^2 \;\leq\; B \sum_{k_1\in K_1; m\in\mathbb{Z}} \mid \alpha_{k_1,m} \mid^2, \tag{5.14}$$

and

$$\parallel \underline{\alpha} \parallel^2 = \sum_{k_1\in K_1; m\in\mathbb{Z}} \mid \alpha_{k_1,m} \mid^2 = \mid G/K_1 \mid \int_{\hat{G}/Ann(K_1)} \sum_{m\in\mathbb{Z}} \mid \alpha_m(\xi) \mid^2 d\xi. \tag{5.15}$$

Using (5.13), (5.14) and(5.15), we obtain

$$\int_{\hat{G}/Ann(K_1)} \sum_{\gamma_1 \in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \alpha_m(\xi)\hat{g}_m(\xi\gamma_1^{-1}) \mid^2 d\xi \leq \mid G/K_1 \mid B \int_{\hat{G}/Ann(K_1)} \sum_{m\in\mathbb{Z}} \mid \alpha_m(\xi) \mid^2 d\xi. \tag{5.16}$$

For $\underline{\beta} \in \ell^2(\mathbb{Z})$, with $\beta_m \neq 0$ for only finitely many $m \in \mathbb{Z}$, choose $\alpha_m(\xi) = \beta_m \rho(\xi)$, where $\rho(\xi) = \sum_{k_1\in K_1} \rho_{k_1} k_1(\xi)$, with $\rho_{k_1} \neq 0$ for only finitely many $k_1 \in K_1$. Thus, we get

$$\int_{\hat{G}/Ann(K_1)} \sum_{\gamma_1 \in Ann(K_1)} \mid \rho(\xi) \mid^2 . \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m \mid^2 d\xi \leq \mid G/K_1 \mid B \parallel \underline{\beta} \parallel^2 \int_{\hat{G}/Ann(K_1)} \mid \rho(\xi) \mid^2 d\xi. \tag{5.17}$$

Since the set of such ρ is dense in $L^2\left(\hat{G}/Ann\,(K_1)\right)$ (because of the fact that K_1 is an orthonormal basis $L^2\left(\hat{G}/Ann\,(K_1)\right)$), we have

$$\sum_{\gamma_1 \in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m \mid^2 \leq \mid G/K_1 \mid B \parallel \underline{\beta} \parallel^2 ,$$

for almost all $\xi \in \hat{G}/Ann\,(K_1)$. Let V be a countable, dense subset of $\ell^2(\mathbb{Z})$ of $\underline{\beta}$'s with $\beta_m \neq 0$ for only finitely many $m \in \mathbb{Z}$, and let $N_1 \subset \hat{G}/Ann\,(K_1)$ be the null set outside of which

$$\sum_{\gamma_1 \in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m \mid^2 \;\leq\; \mid G/K_1 \mid B \parallel \underline{\beta} \parallel^2, \forall\, \underline{\beta} \in V. \tag{5.18}$$

Also, let $N_2 \subset \hat{G}/Ann\,(K_1)$ be a null set outside of which

$$\sum_{m\in\mathbb{Z}} \mid \hat{g}_m(\xi\gamma_1^{-1}) \mid^2 \;\;\leq\;\; \mid G/K_1 \mid .B,\; \forall\, \gamma_1 \in Ann\,(K_1)\,. \tag{5.19}$$

Letting $\underline{\beta} \in \ell^2(\mathbb{Z})$ and $\underline{\beta}^{(M)} \in V$ such that $\underline{\beta}^{(M)} \longrightarrow \underline{\beta}$, and applying Fatou's Lemma, we arrive at

$$\begin{aligned}\sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m \mid^2 \;\;&\leq\;\; \lim_{M\longrightarrow\infty}\inf \sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m^{(M)} \mid^2 \\ &\leq\;\; \mid G/K_1 \mid .B \parallel \underline{\beta} \parallel^2 . \end{aligned} \tag{5.20}$$

Finally, we have

$$\sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m \mid^2 \leq \mid G/K_1 \mid .B \parallel \underline{\beta} \parallel^2 \;\; \forall\, \underline{\beta} \in \ell^2(\mathbb{Z})\;,$$

for almost all $\xi \in \left\{\hat{G}/Ann\,(K_1)\right\} \setminus N$, where $N = N_1 \cup N_2$. Since any element $\nu \in \hat{G}$ can be written as $\nu = \xi\gamma_1$, where $\xi \in \hat{G}/Ann\,(K_1)$ and $\gamma_1 \in Ann\,(K_1)$ and since the first sum in (5.21) is taken over all elements of $Ann\,(K_1)$, we have

$$\sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m \mid^2 \leq \mid G/K_1 \mid .B \parallel \underline{\beta} \parallel^2;\; \underline{\beta} \in \ell^2(\mathbb{Z});\;\; a.e.\;\; \xi \in \hat{G}.$$

□

The following is a generalization of the Theorem 2.4 to $L^2(G)$, for any LCA group G.

Theorem 5.3. *With the same setting as above, the following hold:*

(i) $\{g_{\gamma_1,m}\}$ *is a Bessel sequence with upper bound* B *if and only if for almost all* ξ, $H(\xi)$ *defines a bounded operator from* $\ell^2(\mathbb{Z})$ *into* $\ell^2\left(Ann(K_1)\right)$ *of norm at most* $\sqrt{\mid G/K_1 \mid .B}$.

(ii) $\{g_{\gamma_1,m}\}$ *is a frame for* $L^2(G)$ *with frame bounds* A, B *if and only if*

$$\mid G/K_1 \mid .A\mathbb{I} \leq H(\xi)H^*(\xi) \leq \mid G/K_1 \mid .B\mathbb{I}, \tag{5.21}$$

for almost all ξ, *where* $\mathbb{I}$ *is identity operator on* $\ell^2\left(Ann(K_1)\right)$.

(iii) $\{g_{\gamma_1,m}\}$ *is a tight frame for* $L^2(G)$ *if and only if there is a constant* $c > 0$ *such that*

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m(\xi)}\hat{g}_m(\xi\gamma_1) = c\delta_{\gamma_1,1_G}\;, \quad \gamma_1 \in Ann(K_1)\;, \tag{5.22}$$

for almost all ξ. *In this case, the frame bound is* $A = \dfrac{c}{\mid G/K_1 \mid}$.

(iv) *Two shift-invariant systems* $\{g_{\gamma_1,m}\}$ *and* $\{h_{\gamma_1,m}\}$, *which form Bessel sequences, are dual frames if and only if*

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m(\xi)}\hat{h}_m(\xi\gamma_1) = \mid G/K_1 \mid \delta_{\gamma_1,1_{\hat{G}}}\;, \quad \gamma_1 \in Ann(K_1)\;. \tag{5.23}$$

for almost all ξ.

Proof. For part (iv), it's known that $\{g_{\gamma_1,m}\}$ and $\{h_{\gamma_1,m}\}$ are dual frames if and only if

$$\langle e \mid f\rangle = \sum_{m\in\mathbb{Z}} \sum_{k_1\in Ann(K_1)} \langle e \mid g_{\gamma_1,m}\rangle\langle h_{\gamma_1,m} \mid f\rangle, \quad \forall e,\ f \in\ L^2(G), \tag{5.24}$$

and we have $\rho\,(e,f)\,(x) = \langle T_x e \mid T_x f\rangle = \langle e \mid f\rangle, \quad \forall x \in G/K_1$. Hence the functions $\rho\,(e,f)\,(x)$ and $\langle e \mid f\rangle$ have the same Fourier coefficients in $L^2\,(G/K_1)$, whence

$$\begin{aligned} c_{\gamma_1} &= \frac{1}{\mid G/K_1 \mid} \int_{\hat{G}} \hat{e}(\xi).\overline{\hat{f}(\xi\gamma_1)} \sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1)d\xi \\ &= \langle e \mid f\rangle \delta_{\gamma_1,1_{\hat{G}}} = \delta_{\gamma_1,1_{\hat{G}}} \int_{\hat{G}} \hat{e}(\xi)\overline{\hat{f}(\xi)}d\xi. \end{aligned} \tag{5.25}$$

Since (5.25) holds for all $e \in L^2\,(G)$, we have

$$\overline{\hat{f}(\xi\gamma_1)} \sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1) \;=\; \mid G/K_1 \mid \delta_{\gamma_1,1_{\hat{G}}}\overline{\hat{f}(\xi)},\ a.e.\ \xi \in \hat{G}. \tag{5.26}$$

If $\gamma_1 = 1_{\hat{G}}$, we get

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi) \;=\; \mid G/K_1 \mid, \quad a.e.\ \xi \in \hat{G}. \tag{5.27}$$

If $\gamma_1 \neq 1_{\hat{G}}$, then $\gamma_1\xi \neq \xi, \quad \forall \xi \in \hat{G}$. Since $\hat{G}$ is Hausdorff, there exists an open neighbourhood, $\mathcal{O}_{\gamma_1\xi}$ of $\gamma_1\xi$, such that $\xi \notin \mathcal{O}_{\gamma_1\xi}$. By taking $\hat{f}(s) = \chi_{\mathcal{O}_{\gamma_1\xi}}(s)$, the charactiristic function of $\mathcal{O}_{\gamma_1\xi}$, and using this function in (5.26), we obtain

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1) \;=\; 0, \tag{5.28}$$

for almost all $\xi \in \hat{G}$ and for all $\gamma_1 \neq 1_{\hat{G}}$. Thus,

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1) = \mid G/K_1 \mid \delta_{\gamma_1,1_{\hat{G}}}, \quad \gamma_1 \in Ann(K_1), \tag{5.29}$$

for almost all $\xi \in \hat{G}$. For the opposite implication let us assume that (2.6) holds. It follows that the function $\rho\,(e,f)$ and $\langle e \mid f\rangle$ have the same Fourier coefficients and then $\rho\,(e,f)\,(x) = \langle e \mid f\rangle$, $\forall x \in G$. By taking $x = 1_G$, we obtain (5.24).

For the first part of (iii), we use (iv) and Lemma 2.1. For the second part, we use (ii) and the fact that the γ_1-st row and γ_2-nd column of $H(\xi)H^*(\xi)$ are $\sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\overline{\hat{g}_m(\xi\gamma_2^{-1})} = c\delta_{\gamma_1,\gamma_2}$, to get $A = \frac{c}{|G/K_1|}$.

For part (ii), if $\sum_{\gamma_1\in Ann(K_1),m\in\mathbb{Z}} \mid \langle f, g_{\gamma_1,m}\rangle \mid^2 \leq B \parallel f \parallel^2$, then by virtue of Proposition 5.2, $H(\xi)H^*(\xi) \leq \mid G/K_1 \mid B\mathbb{I}$. Let f be an element in $L^2(G)$ with $\hat{f}$ compactly supported and let $m \in \mathbb{Z}$. The series

$$\sum_{\gamma_1\in Ann(K_1)} \hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1) \tag{5.30}$$

defines a function in $L^2\left(\hat{G}/Ann(K_1)\right)$. Since K_1 is an orthogonal basis of $L^2\left(\hat{G}/Ann(K_1)\right)$, it follows that

$$\sum_{\gamma_1\in Ann(K_1)} \hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1) = \sum_{k_1\in K_1} c_{k_1,m}\xi(k_1), \tag{5.31}$$

where

$$\begin{aligned}
c_{k_1,m} &= \mid G/K_1 \mid \int_{\hat{G}/Ann(K_1)} \sum_{\gamma_1 \in Ann(K_1)} \overline{\xi(k_1)} \hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1)d\xi \\
&= \mid G/K_1 \mid \int_{\hat{G}} \overline{\xi(k_1)}\hat{g}_m(\xi)\hat{f}^*(\xi)d\xi \\
&= \mid G/K_1 \mid \int_{\hat{G}} \mathcal{F}\left(T_{k_1}g_m\right)(\xi)\mathcal{F}\left(f^*\right)(\xi)d\xi \\
&= \mid G/K_1 \mid \int_{\hat{G}} g_m(\xi {k_1}^{-1})f^*(\xi)d\xi \\
&= \mid G/K_1 \mid \langle g_{k_1,m}, f\rangle \ . \qquad (5.32)
\end{aligned}$$

Thus,

$$\int_{\hat{G}/Ann(K_1)} \mid \sum_{\gamma_1 \in Ann(K_1)} \hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1) \mid^2 d\xi = \mid G/K_1 \mid \sum_{k_1 \in K_1} \mid \langle g_{k_1,m}, f\rangle \mid^2 . \qquad (5.33)$$

Let $\xi \in \hat{G}$ and set $\underline{\hat{f}}(\xi) = \{\underline{f}(\xi\gamma_1^{-1})\}_{\gamma_1 \in K_1}$. We then have

$$\begin{aligned}
\int_{\hat{G}/Ann(K_1)} \parallel \underline{\hat{f}}(\xi) \parallel^2 d\xi &= \int_{\hat{G}/Ann(K_1)} \sum_{\gamma_1 \in Ann(K_1)} \mid \hat{f}(\xi\gamma_1^{-1}) \mid^2 d\xi \\
&= \int_{\hat{G}} \mid \hat{f}(\xi) \mid^2 d\xi \\
&= \parallel f \parallel^2 , \qquad (5.34)
\end{aligned}$$

while use of (5.33) yields

$$\begin{aligned}
&\int_{\hat{G}/Ann(K_1)} \parallel H^*(\xi)\underline{\hat{f}}(\xi) \parallel^2 d\xi \\
&= \sum_{m\in\mathbb{Z}} \int_{\hat{G}/Ann(K_1)} \mid \sum_{\gamma_1 \in Ann(K_1)} \hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1) \mid^2 d\xi \\
&= \mid G/K_1 \mid \sum_{m\in\mathbb{Z}} \sum_{k_1 \in K_1} \mid \langle g_{k_1,m}, f\rangle \mid^2 . \qquad (5.35)
\end{aligned}$$

Let $\hat{\rho} \in L^2\left(\hat{G}/Ann(K_1)\right)$ and let $\underline{\beta} \in \ell(\mathbb{Z})$ with $\beta_k \neq 0$ for only finitely many $k \in \mathbb{Z}$. Take any section $\sigma : \hat{G}/Ann(K_1) \longrightarrow \hat{G}$ and let $\hat{\rho}_\sigma$ be the function defined on $\hat{G}$ with support in $V_\sigma = \sigma\left(\hat{G}/Ann(K_1)\right)$ and such that $\hat{\rho}_\sigma(\xi) = \hat{\rho}(\pi(\xi))$ if $\xi \in V_\sigma$ and $\hat{\rho}_\sigma(\xi) = 0$ otherwise. It is then easy to see that

$$\int_{\hat{G}/Ann(K_1)} \mid \hat{\rho}(\xi) \mid^2 d\xi = \int_{V_\sigma} \mid \hat{\rho}_\sigma(\xi) \mid^2 d\xi. \qquad (5.36)$$

Assuming that $\mid Ann(K_1) \mid = \infty$, let us write $Ann(K_1)$ in the form

$$Ann(K_1) = \{\gamma_{1,k} : k \in \mathbb{Z}\} . \qquad (5.37)$$

Define a function $\hat{f}$ on $\hat{G}$ by $\hat{f}(\xi) = \beta_k\hat{\rho}_\sigma(\xi\gamma_{1,k})$, where k is such that $\xi\gamma_{1,k} \in V_\sigma$ and let $\underline{\hat{f}}(\xi) = \{\beta_k\hat{f}(\xi\gamma_{1,k}^{-1})\}_{k\in\mathbb{Z}}$. It is then easy to see that

$$\underline{\hat{f}}(\xi) = \underline{\beta}\hat{\rho}_\sigma(\xi) \ . \qquad (5.38)$$

Furthermore, from (5.34) and (5.35) and the first (5.21), we have

$$
\begin{aligned}
(5.39)\qquad & \int_{\hat{G}/Ann(K_1)} \| H^*(\xi)\underline{\hat{f}}(\xi) \|^2 \, d\xi \\
&= \int_{\hat{G}/Ann(K_1)} | \hat{\rho}(\xi) |^2 \| H^*(\xi)\underline{\beta} \|^2 \, d\xi \\
&= | G/K_1 |. \sum_{m\in\mathbb{Z}} \sum_{k_1\in K_1} | \langle g_{k_1,m}, f\rangle |^2 \\
&\geq | G/K_1 | .A \| f \|^2 \\
&= | G/K_1 | .A \int_{\hat{G}/Ann(K_1)} \| \underline{f}(\xi) d\xi \|^2 \\
&= | G/K_1 | .A \int_{\hat{G}/Ann(K_1)} \sum_{k\in\mathbb{Z}} | \beta_k.\hat{\rho}(\xi) |^2 \, d\xi \\
&= | G/K_1 | .A \| \underline{\beta} \|^2 \int_{\hat{G}/Ann(K_1)} | \hat{\rho}(\xi) |^2 \, d\xi.
\end{aligned}
$$

Letting $\hat{\rho}$ run over all of $L^2\left(\hat{G}/Ann(K_1)\right)$, we obtain

$$
(5.40)\qquad \| H^*(\xi)\underline{\beta} \|^2 \geq | G/K_1 | .A \| \underline{\beta} \|^2 ,
$$

for almost all $\xi \in \hat{G}/Ann(K_1)$, where the null set involved in (5.40) may depend on $\underline{\beta}$. Let V be a countable dense set of $\underline{\beta}$'s in $\ell^2(\mathbb{Z})$ such that $\beta_k \neq 0$ for only finitely many $k \in \mathbb{Z}$ and let $N_1 \subset \hat{G}/Ann(K_1)$ be a null set such that

$$
(5.41)\qquad \| H^*(\xi)\underline{\beta} \|^2 \geq | G/K_1 | .A \| \underline{\beta} \|^2;\ \beta \in V,\ \xi \in \hat{G}/Ann(K_1)/N_1.
$$

Also, let $N_2 \subset \hat{G}/Ann(K_1)$ be a null set such that

$$
(5.42)\qquad \| H^*(\xi)\underline{\beta} \|^2 \leq | G/K_1 | .B \| \underline{\beta} \|^2;\ \underline{\beta} \in \ell^2(\mathbb{Z}),\ \xi \in \hat{G}/Ann(K_1)/N_2.
$$

Take $\xi \in \left(\hat{G}/Ann(K_1)\right) - (N_2 \cup N_2)$, $\underline{\beta} \in \ell^2(\mathbb{Z})$ and $\underline{\beta}^{(M)} \in V$ such that $\beta^{(M)} \longrightarrow \underline{\beta}$. Then, from (5.41) and (5.42), we conclude that

$$
\begin{aligned}
\| H^*(\xi)\underline{\beta} \|^2 &= \lim_{M\to\infty} \| H^*(\xi)\underline{\beta}^{(M)} \|^2 \geq | G/K_1 | .A \lim_{M\to\infty} \| \underline{\beta}^{(M)} \|^2 \\
(5.43)\qquad &= | G/K_1 | .A \| \underline{\beta} \|^2 .
\end{aligned}
$$

This completes the proof of the implication "⇒" of part (ii). To prove the opposite implication, let $f \in L^2(G)$ such that $\hat{f}$ is compactly supported in $\hat{G}$. Then (5.34) and (5.35), imply

$$
A \| f \|^2 \leq \sum_{\gamma_1\in Ann(K_1), m\in\mathbb{Z}} | \langle f, g_{\gamma_1,m}\rangle |^2 \leq B \| f \|^2 .
$$

□

6. Conclusion

We have concentrated in the present paper on working out the essential mathematical results, generalizing some earlier theoretical results based on $G = \mathbb{R}^d$. The cases $G = \mathbb{R}^d$ and $G = \mathbb{T}^d$ are of immediate physical interest. In a future publication, we intend to dwell

upon some physical applications, which make use of coherent states and frames built out of the generalized Weyl-Heisenberg groups discussed here. In particular, coherent states and frames built on the cotangent bundle of the torus in d-dimensions could find useful applications in atomic physics. While the p-adic groups are also LCA groups, we are unaware of any obvious physical applications of frames or coherent states built out of them. Also, since the Haar measure plays a crucial role in the development of the results presented above, we do not see any immediate way of extending these results to infinite dimensional groups. In a subsequent publication we also intend to present results generalizing the Walnut representation [2] of the frame operator and its representation in frequency domain to the present setting.

Acknowledgments

The authors would like to thank H. Feichtinger, G. Kutinyok and Z. Shen for useful discussions. They also benefitted from grants from the Natural Sciences and Engineering Research Council (NSERC) of Canada and the Fonds de recherche sur la nature et les technologies (FQRNT), Québec.

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